Coarse Resistance Tree Methods For Stochastic Stability Analysis

Holly Borowski, Jason R. Marden, David S. Leslie, and Eric W. Frew

Abstract— **Emergent behavior in natural and manmade systems can often be characterized by the limiting distribution of a class of Markov processes termed regular perturbed processes. Resistance trees have gained popularity as a computationally efficient way to characterize the support of the limiting distribution; however, there are three main limitations of this approach. First, it requires finding a minimum weight spanning tree for each state in a potentially large state space. Second, perturbations to transition probabilities must decay at an exponentially smooth rate. Lastly, the approach is shown to hold purely in the context of finite Markov chains. In this paper we seek to address these limitations by developing new tools for characterizing the limiting distribution. First, we provide necessary conditions for stochastic stability via a coarse, and less computationally intensive, state space analysis. Next, we identify necessary conditions for stochastic stability when smooth convergence requirements are relaxed. Finally, we establish similar tools for stochastic stability analysis in Markov chains over a continuous state space.**

I. Introduction

Markov chains are used to model dynamical processes in engineering and social sciences [4], [5], [10], [19], [20]. Emergent behavior of such systems can be characterized by analyzing the stationary distributions of the governing Markov chain. Providing a precise characterization of the stationary distributions can be computationally prohibitive when the state space is large. Furthermore, in situations where the stationary distribution is not unique, characterizing the impact of initial conditions can be challenging.

One approach for addressing these issues is to consider small perturbations to the nominal process described above. We introduce small perturbations to the nominal transitional probabilities so that (i) the resulting perturbed Markov chain has a unique stationary distribution and (ii) this perturbed Markov chain will closely approximate the nominal Markov chain when the size of the perturbations goes to zero. As perturbations are driven to zero, the unique stationary distribution associated with the perturbed Markov chain is a stationary distribution of the nominal process. Significant research has focused on deriving computationally effective ways to characterize these limiting distributions, whose support is termed the *stochastically stable states*, for specific perturbation models [6], [11], [12], [27].

Regular perturbed processes [12], [27] are a type of perturbation model that is actively studied in both engineering and social sciences. They have been used in the social sciences to model natural human tendencies, e.g., mistakes or experimentation [1], [3], [12]–[14], [21], [27]. In engineering systems, regular perturbed processes have been used to prescribe distributed control laws that guarantee desired equilibrium selection [2], [7], [9], [15], [17], [26], [28], [29]. For example, [28] introduces a distributed learning algorithm which is modeled as a regularly perturbed process that guarantees convergence to a pure Nash equilibrium in virtually any game where such an equilibrium exists.

The structure of the perturbations in regular perturbed processes has been exploited to develop computationally efficient mechanisms for determining the stochastically stable states. Building off the seminal work of [8], the authors in [12], [27] demonstrate that the stochastically stable states in any regular perturbed process can be characterized by finding a minimum weight spanning tree rooted at each state in the state space. These graph theoretic tools, termed *resistance trees*, significantly reduce the computation associated with characterizing the limiting behavior of perturbed Markov chains when compared to traditional eigenvalue/eigenvector methods. However, these methods are only effective in characterizing the support of the limiting distribution.

Resistance tree methods have limitations. First, determining the minimum weight spanning trees over a large state space can be computationally prohibitive. Many recent developments in distributed learning introduce auxiliary state variables to the individual agents as a coordinating mechanism to drive the system to specific classes of equilibria [16], [18], [23], [28], heightening the sensitivity to this limitation. The addition of a single binary state variable to a single agent doubles the size of the state space, which can enhance the challenge in determining minimum weight spanning trees. A second limitation of resistance tree methods is their dependence on specific perturbation models. This dependence can impose constraining assumptions on the prescribed system dynamics, e.g., the assumptions of Theorem 4.3 in [15]. Lastly, existing research extending resistance tree methods from the finite state space to continuous state space are often highly specialized which limits their applicability to general system analysis and design [22], [24]–[26].

This work seeks to address the limitations highlighted above. In Section III, we derive sufficient conditions for stochastic stability which entails a graph theoretic analysis over a partitioned state space. Then, in Section IV we provide a more general perturbations model and establish necessary conditions for stochastic stability. In Section V we establish similar results for Markov processes over a continuous state space. We end with an illustrative example in Section VI.

II. Preliminaries: resistance trees for regular perturbed processes

Let P^0 be the transition matrix for a stationary Markov chain defined over finite state space X, where, for $x, y \in X$,

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P^0_{xy} is the probability of transitioning in a single step from state x to state y. Consider a family of processes over X in which transitions occur with high probability according to P^0 and with low probability in some way that does not follow P^0. We model this family of perturbations to P^0 by the stationary Markov chains with transition matrices $\{P^\varepsilon\}$ and transition probabilities $\{P^\varepsilon_{xy}\}$, where $\varepsilon \in (0, a]$, $a > 0$ parameterizes perturbations to the original process P^0.

We focus on the class of *regular perturbed processes* introduced in [27]. The Markov chain P^ε is a regular perturbed process (RPP) if:

(1) P^ε is aperiodic and irreducible for all $\varepsilon \in (0, a]$
(2) $\lim_{\varepsilon \to 0^+} P^\varepsilon_{xy} = P^0_{xy},\ \forall x, y \in X$
(3) If $P^\varepsilon_{xy} > 0$ for some $\varepsilon \in (0, a]$, then there exists $r(x, y) \geq 0$ such that

$$0 < \lim_{\varepsilon \to 0^+} \frac{P^\varepsilon_{xy}}{\varepsilon^{r(x,y)}} < \infty, \tag{1}$$

where $r(x, y)$ is the *resistance* of the transition from x to y.

Because P^ε is aperiodic and irreducible, it has a unique stationary distribution, μ^ε. whereas P^0, may have multiple stationary distributions. For an RPP, $\lim_{\varepsilon \to 0^+} \mu^\varepsilon = \mu^0$ exists and is unique, where μ^0 is a stationary distribution of P^0 [27]. That is, P^ε selects the stationary distribution μ^0 of P^0 as $\varepsilon \to 0$. A state $x \in X$ is *stochastically stable* if it belongs to the support of this distribution, i.e., $\mu^0_x > 0$ [6].

Full state space analysis: As in [27] we define a weighted directed graph, $G = (X, E)$. For $x, y \in X$, $(x, y) \in E$ if and only if $P^\varepsilon_{xy} > 0$ for all $\varepsilon > 0$. Edge (x, y) is weighted by resistance $r(x, y)$. A path in G from x to y is a sequence of vertices $\mathcal{P}_{x \to y} = (x = x_0, x_1, x_2, \ldots, x_k = y)$, with no repeated vertices and with $(x_{j-1}, x_j) \in E, \forall j \in \{1, \ldots, k\}$. Irreducibility of P^ε guarantees that there is a path in G between any two states in X; hence G is connected. A tree $T_x = (X, E_{T_x})$ is said to be an x-tree if it spans G and there is a unique path in T_x from y to x, $\forall y \in X, y \neq x$. The resistance of T_x is

$$r(T_x) = \sum_{(x,y) \in E_{T_x}} r(x, y). \tag{2}$$

Let $\mathbf{T}_x$ be the set of all x-trees. The quantity

$$\gamma(x) := \min_{T \in \mathbf{T}_x} r(T_x) \tag{3}$$

is the *stochastic potential* of state x, and

$$\mu^0_x > 0 \iff \gamma(x) \leq \gamma(y),\ \forall y \in X. \tag{4}$$

That is, a state is stochastically stable if and only if has minimum stochastic potential over all states in X [27].

Analysis over recurrent communication classes: The *recurrent communication classes* [27] of P^ε are the disjoint subsets, $H_1, H_2, \ldots, H_f$ which are the recurrent classes of the unperturbed process P^0. To characterize the support of the limiting stationary distribution, μ^0, [27] builds a graph over the recurrent classes of X. Determining the weight of an edge from recurrent class H_i to class H_j requires finding a lowest resistance path from H_i to H_j, i.e.,

$$r(H_i, H_j) = \min_{\mathcal{P} \in \mathbf{P}_{i \to j}} r(\mathcal{P}), \tag{5}$$

where $\mathbf{P}_{i \to j}$ denotes the set of all paths originating in set H_i and ending in set H_j, and $r(\mathcal{P})$ is the sum of the resistances of the edges in $\mathcal{P}$. Computing such minimum resistance paths is often difficult. The stochastic potential of a recurrent class is the total resistance of the lowest resistance tree rooted at that class. Young [27] proves that the stochastic potential of a class equals the stochastic potential of the states it contains; an analysis similar to (4) can be conducted to evaluate which recurrent classes are stochastically stable.

III. Coarse analysis of RPPs

Motivated by the first limitation highlighted above, we perform a coarse analysis of RPPs to provide a more computationally tractable method of determining stochastic stability. We partition the state space and define resistances between partitions in a way that leads to bounds on the stochastic potential of states within each partition. We use these bounds to establish sufficient conditions for a partition to contain a stochastically stable state.

A. Notation and definitions

Let P^ε be an RPP over finite state space X, with graph $G = (X, E)$. We shift focus from identifying stochastically stable states to identifying sets which contain them. A set $U \subseteq X$ is referred to as *stochastically stable* if $\exists x \in U$ such that $\lim_{\varepsilon \to 0^+} \mu^\varepsilon_x = \mu^0_x > 0$.

A coarse representation of a partitioned space: Let $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ be a partitioning of state space X, and let $\mathcal{G} = (\{1, \ldots, m\}, \mathcal{E})$ be a weighted directed graph over the partition indices with $(i, j) \in \mathcal{E}$ if and only if there exist $x \in X_i,\ y \in X_j$ such that $(x, y) \in E$. Edge $(i, j) \in \mathcal{E}$ is assigned weight

$$r_{ij} := \min_{\substack{x \in X_i \\ y \in X_j}} r(x, y), \tag{6}$$

the lowest resistance of any single step transition from X_i to X_j. Minimizing over single step transitions is simpler than minimizing over all paths between classes as in (5). Denote a spanning tree of $\mathcal{G}$ rooted at j by $\mathcal{T}_{X_j}$ and refer to it as a j-tree. Define the stochastic potential of partition X_j as

$$\gamma(X_j) := \min_{\mathcal{T}_{X_j} \in \tau_j} r(\mathcal{T}_{X_j}), \tag{7}$$

where τ_j denotes the set of spanning trees of $\mathcal{G}$ with root j.

A fine representation within partitionings: Let $G|_{X_i} = (X_i, E_i)$ be the subgraph of G restricted to X_i. The notation $T_{x|X_i}$ refers to a minimum resistance spanning tree of $G|_{X_i}$ rooted at state $x \in X_i$. Define

$$\rho^{\mathrm{U}}_i := \max_{x \in X_i} r(T_{x|X_i}) \qquad \rho^{\mathrm{L}}_i := \min_{x \in X_i} r(T_{x|X_i}) \tag{8}$$

to be the highest and lowest minimum resistance spanning trees of $G|_{X_i}$, respectively. These quantities will be used to construct minimum resistance trees over the full state space.

Notation summary:

<u>Full state space analysis</u>

- X - The full state space
- G - A graph over X, edges defined by transitions in P^ε
- $r(x, y)$ - Resistance of the transition from x to y
- T_x, an x-tree - A spanning tree of G rooted at $x \in X$
- $r(T_x)$ - Total resistance of all the edges in T_x

<u>Coarse state space analysis</u>

- $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ - A partitioning of state space X

- $\mathcal{G}$ - A graph over the partitions in $\mathcal{X}$
- r_{ij} - Lowest single step resistance from X_i to X_j, edge weight of $(i,j) \in \mathcal{G}$
- $\mathcal{T}_{X_j}$, a j-tree - A spanning tree of $\mathcal{G}$ rooted at partition j

Fine representation within partitionings

- $G|_{X_i}$ - Graph G restricted to the set X_i
- $T_{x|X_i}$ - Minimum resistance spanning tree of $G|_{X_i}$ rooted at $x \in X_i$
- ρ_i^{U} - Resistance of the minimum resistance tree $T_i(x)$, maximized over $x \in X_i$
- ρ_i^{L} - Resistance of the minimum resistance tree $T_i(x)$, minimized over $x \in X_i$

B. Stochastic stability for arbitrary partitionings

We consider an arbitrary state space partitioning, requiring only that each subgraph, $G|_{X_i}$, contains a path between each pair of states in $G|_{X_i}$.

Theorem 1: Let P^ε be an RPP over state space X, and let $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ partition X such that, for any $x, y \in X_i$ and for all $\varepsilon \in (0,1)$ there exists a sequence of positive probability transitions of P^ε beginning at x and ending at y. Partition X_j is stochastically stable if

$$\gamma_a^{\mathrm{U}}(X_j) \leq \gamma_a^{\mathrm{L}}(X_i), \quad \forall i \in \{1, 2, \ldots, m\} \tag{9}$$

where

$$\gamma_a^{\mathrm{L}}(X_i) := \gamma(X_i) + (|X| - m) \min_{x,y\in X} r(x,y)$$
$$\gamma_a^{\mathrm{U}}(X_j) := \gamma(X_j) + \min_{x\in X_j} \sum_{\substack{x\in X_j \\ \setminus\{x_j\}}} \min_{\mathcal{P}\in\mathbf{P}_{x\to x_j}} r(\mathcal{P}) + \sum_{i\neq j} \rho_i^{\mathrm{U}}.$$

Proof: We bound the stochastic potential $\gamma(x_j)$, $x_j \in X_j$, by

$$\gamma_a^{\mathrm{L}}(X_j) \leq \gamma(x_j) \leq \gamma_a^{\mathrm{u}}(X_j). \tag{10}$$

To establish the theorem, we apply (4) and the above lower and upper bounds on a state's stochastic potential.

Lower bounding the stochastic potential: To lower bound $\gamma(x_j)$, we determine the lowest possible resistance of an x_j-tree, $T_{x_j} = (X, E_{T_{x_j}})$. For all $i \neq j$, each partition X_i has at least one edge of T_{x_j} leaving it. Consider the graph $\mathcal{G}'$ with vertices $\{1, \ldots, m\}$ and an edge (i,k) if and only if there exist $x \in X_i,\ y \in X_k$ with $(x,y) \in E_{T_{x_j}}$, with resistance

$$r_{ik} = \min_{\substack{(x,y)\in T_{x_j} \text{ s.t.} \\ x\in X_i,\, y\in X_j}} r(x,y).$$

Since there was a path in T_{x_j} from each state in X to x_j, this graph must have a subtree rooted at j; denote one such subtree by $\mathcal{T}_{X_j}$. Clearly $\gamma(X_j) \leq r(\mathcal{T}_{X_j})$. The resistance of each edge in T_{x_j} not corresponding to an edge in $\mathcal{T}_{X_j}$ can be lower bounded by $\min_{x,y\in X} r(x,y)$. There are $|X| - m$ such edges in T_{x_j}; hence $\gamma_a^{\mathrm{L}}(X_j) = \gamma(X_j) + (|X| - m)\min_{x,y\in X} r(x,y)$ lower bounds $r(T_{x_j})$ for any x_j-tree T_{x_j}, and thus also lower bounds $\gamma(x_j)$.

Upper bounding the stochastic potential: To upper bound the stochastic potential of a state x_j, we construct a spanning tree T_{x_j} of G rooted at x_j with resistance $r(T_{x_j}) \leq \gamma_a^{\mathrm{U}}(x_j)$. Let $\mathcal{T}_{X_j} = (\{1, \ldots, m\}, \mathcal{E}_{\mathcal{T}_{X_j}})$ be a j-tree in $\mathcal{G}$ with resistance $\gamma(X_j)$. Begin with vertex set X and add edges to construct $T_{x_j} = (X, E_{T_{x_j}})$ as follows:

(1) For each $(i,k) \in \mathcal{E}_{\mathcal{T}_{X_j}}$, choose $x \in X_i$ and $y \in X_k$ with $r(x,y) = r_{ik}$, and add (x,y) to $E_{T_{x_j}}$. These states exist by definition of $\mathcal{G}$. This adds resistance $\gamma(X_j)$ to T_{x_j}.

(2) For all $i \neq j$, add the edges of minimum resistance tree $T_{x_i|X_i}$, where $x_i \in X_i$ is the vertex with an outgoing edge (x_i, y) from the previous step. For each $i \neq j$ this adds exactly one subtree with resistance at most ρ_i^{U}, adding at most resistance $\sum_{i\neq j} \rho_i^{\mathrm{U}}$ to T_{x_j}.

(3) For each $x \in X_j$, $x \neq x_j$, add the edges of a minimum resistance path $\mathcal{P}_{x\to x_j}$ and eliminate any redundant edges in T_{x_j}. This adds at most resistance $\sum_{x\in X_j\setminus\{x_j\}} \min_{\mathcal{P}\in\mathbf{P}_{x\to x_j}} r(\mathcal{P})$.

We have constructed a tree over vertex set X rooted at x_j with resistance at most

$$\gamma(X_j) + \sum_{x\in X_j\setminus\{x_j\}} \min_{\mathcal{P}\in\mathbf{P}_{x\to x_j}} r(\mathcal{P}) + \sum_{i\neq j} \rho_i^{\mathrm{U}},$$

so, for all $x_j \in X_j$,

$$\gamma(x_j) \leq \gamma_a^{\mathrm{U}}(X_j) = \gamma(X_j) + \min_{x\in X_j} \sum_{\substack{x\in X_j \\ \setminus\{x_j\}}} \min_{\mathcal{P}\in\mathbf{P}_{x\to x_j}} r(\mathcal{P}) + \sum_{i\neq j} \rho_i^{\mathrm{U}}. \tag{11}$$

Completing the proof: The theorem follows by applying (4) and the above bounds on a state's stochastic potential. ■

C. Stochastic stability for minimal resistance partitionings

Often the structure of the problem considered results in natural structure on the state space partitioning, allowing us to establish useful sufficient conditions for stochastic stability. We introduce the following definition:

Definition 1: A *minimal resistance partitioning* of state space X is a partitioning $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ such that for any $i \in \{1, \ldots, m\}$ and for any $x, y \in X_i, z \notin X_i$, there exists a path $\mathcal{P}_{x\to y}$ in $G|_{X_i}$ with $r(P_{x\to y}) \leq r(x,z)$.

In a minimal resistance partitioning it is easier to transition between states within the partition than to exit. We begin with a lemma on their structure.

Lemma 1: For any minimum resistance x_j-tree in T_{x_j} in G, minimal resistance partitioning $\mathcal{X}$, and $X_i \in \mathcal{X}$, there exists a tree T'_{x_j} rooted at x_j of equal resistance such that in T'_{x_j} there is at most one edge leaving each X_i for $i \neq j$ and no edge leaving X_j.

The proof follows a tree manipulation argument and its details are omitted for brevity.

Theorem 2: Given an RPP P^ε over state space X and a minimal resistance partitioning $\mathcal{X} = \{X_1, \ldots, X_m\}$, partition X_j is stochastically stable if

$$\gamma_m^{\mathrm{U}}(X_j) \leq \gamma_m^{\mathrm{L}}(X_i), \quad \forall i \in \{1, \ldots, m\} \tag{12}$$

where

$$\gamma_{\mathrm{m}}^{\mathrm{L}}(X_i) := \gamma(X_i) + \sum_{k=1}^{m} \rho_k^{\mathrm{L}}, \quad \gamma_{\mathrm{m}}^{\mathrm{U}}(X_j) := \gamma(X_j) + \rho_j^{\mathrm{L}} + \sum_{i\neq j} \rho_i^{\mathrm{U}}.$$

The stochastically stable state in a stochastically stable partition X_j is given by $x_j = \arg\min_{x\in X_j}\ r(T(x))$.

Proof: Let P^ε be an RPP over state space X and let $\mathcal{X} = \{X_1, \ldots, X_m\}$ be a minimal resistance partitioning of X. We bound the stochastic potential of $x_j \in X_j$ as

$$\gamma_{\mathrm{m}}^{\mathrm{L}}(X_j) \leq \gamma(x_j) \leq \gamma_{\mathrm{m}}^{\mathrm{U}}(X_j). \tag{13}$$

Applying Equations (13) and (4) establishes the theorem.

Lower bounding the stochastic potential: Let $T_{x_j} = (X, E_{T_{x_j}})$ be a spanning tree of G rooted at x_j with resistance $\gamma(x_j)$, and let $\mathcal{X} = \{X_1, \ldots, X_m\}$ be a minimal resistance partitioning of X. By Lemma 1, we may assume that for all $i \neq j$, there is exactly one edge in T_{x_j} leaving X_i, and that there are no edges leaving X_j. Given vertices $\{1, \ldots, m\}$ construct a j-tree, $\mathcal{T}_{X_j}$ by adding edge (i,k) for $i, k \in \{1, \ldots, m\}$ if an only if there exist $x \in X_i$ and $y \in X_k$ such that $(x,y) \in E_{T_{x_j}}$. Clearly $r(T_{x_j}) \geq \gamma(X_j)$. Since there is only one edge in T_{x_j} leaving any partition X_i, the subgraph $T_{x|X_i}$ is a spanning tree of $G|_{X_i}$ with resistance greater than or equal to ρ_i^{L}. Combining,

$$\gamma_{\mathrm{m}}^{\mathrm{L}} = \gamma(X_j) + \sum_{i=1}^{m} \rho_i^{\mathrm{L}} \leq r(\mathcal{T}) + \sum_{i=1}^{m} \rho_i^{\mathrm{L}} \leq \gamma(x_j). \quad (14)$$

Upper bounding the stochastic potential: Establishing the upper bound

$$\gamma(x_j) \leq \gamma(X_j) + \rho_j^{\mathrm{L}} + \sum_{i \neq j} \rho_i^{\mathrm{U}} = \gamma_{\mathrm{m}}^{\mathrm{U}}$$

follows a tree construction argument similar to the proof of Theorem 1. Details are omitted for brevity.

Completing the proof: The first part of the theorem follows directly from (4) and the above upper and lower bounds. From step (3) in construction of T_{x_j} for upper bounding the stochastic potential of $x_j \in X_j$, a state which satisfies

$$x_j \in \arg\min_{x \in X_j} r(T_x) \quad (15)$$

for stochastically stable X_j is itself stochastically stable. ■

We now examine a third type of partitioning which allows us to fully characterize the stochastically stable states.

Definition 2: A *zero resistance partitioning* is a partitioning $\mathcal{X} = \{X_1, \ldots, X_m\}$ of X such that, $\forall x, y \in X_i,\ i \in \{1, \ldots m\}$ there exists a path $\mathcal{P}_{x \to y}$ with $r(\mathcal{P}_{x \to y}) = 0$.

Corollary 1 follows immediately from the fact that $\rho_i^{\mathrm{U}} = \rho_i^{\mathrm{L}} = 0,\ \forall i \in \{1, \ldots, m\}$ for a zero resistance partitioning.

Corollary 1: Given an RPP P^ε over state space X and a zero resistance partitioning $\mathcal{X} = \{X_1, \ldots, X_m\}$,

$$\gamma(X_j) = \gamma(x_j), \qquad \forall x_j \in X_j, j \in \{1, \ldots m\}. \quad (16)$$

A unique partition X_j is stochastically stable if and only if

$$\gamma(X_j) \leq \min_{i \neq j} \gamma(X_i) \quad (17)$$

IV. Irregular perturbed processes

We now relax the requirement that P^ε must be an RPP and establish necessary conditions for stochastic stability.

A. Resistance trees for irregular perturbed processes

Definition 3: An *irregular perturbed process* (IPP), P^ε of P^0 over the finite state space X is a process parameterized by ε which satisfies the first two properties of an RPP from Section II, along with the following:

(3I) For all $x, y \in X$ such that $P_{xy}^\varepsilon > 0$ for some $\varepsilon > 0$, there exist positive $C^{\mathrm{L}}(x,y)$, $C^{\mathrm{U}}(x,y)$, $r^{\mathrm{L}}(x,y)$, $r^{\mathrm{U}}(x,y)$ such that

$$C^{\mathrm{U}}(x,y)\varepsilon^{r^{\mathrm{U}}(x,y)} \leq P_{xy}^\varepsilon \leq C^{\mathrm{L}}(x,y)\varepsilon^{r^{\mathrm{L}}(x,y)} \quad (18)$$

for all $\varepsilon > 0$. Note, $r^{\mathrm{L}}(x,y) \leq r^{\mathrm{U}}(x,y)$; we call $r^{\mathrm{L}}(x,y)$ and $r^{\mathrm{U}}(x,y)$ the *lower* and *upper resistance* of transition $x \to y$ respectively.

Recall that for an RPP Equation (1) holds so that P^ϵ satisfies (18) if we set $r^{\mathrm{L}}(x,y) = r^{\mathrm{U}}(x,y) = r(x,y)$ and choose $C^{\mathrm{L}}(x,y)$ and $C^{\mathrm{U}}(x,y)$ accordingly. Thus, this property is a generalization of property (3) of an RPP.

For a tree $T = (X, E_T)$,

$$r^{\mathrm{L}}(T) := \sum_{(x,y) \in E_T} r^{\mathrm{L}}(x,y) \quad \text{and} \quad r^{\mathrm{U}}(T) := \sum_{(x,y) \in E_T} r^{\mathrm{U}}(x,y).$$

Define lower and upper stochastic potentials of state x by

$$\gamma^{\mathrm{L}}(x) := \min_{T \in \mathbf{T}_x} r^{\mathrm{L}}(T) \quad \text{and} \quad \gamma^{\mathrm{U}}(x) := \min_{T \in \mathbf{T}_x} r^{\mathrm{U}}(T).$$

B. Stochastic stability for irregular perturbed processes

The following theorem establishes necessary conditions for stochastic stability in an IPP.

Theorem 3: Let P^ε be an IPP of P^0 over state space X, and let μ^ε be its unique stationary distribution. Define $\gamma_{\min}^{\mathrm{U}} = \min_{x \in X} \gamma^{\mathrm{U}}(x)$. For any $x \in X$,

$$\gamma^{\mathrm{L}}(x) > \gamma_{\min}^{\mathrm{U}} \Rightarrow \lim_{\varepsilon \to 0^+} \mu_x^\varepsilon = 0, \quad (19)$$

i.e., x is not stochastically stable.

Proof: Suppose P^ε is an IPP of P^0 over X. Choose $x \in X$ with $\gamma^{\mathrm{L}}(x) > \gamma_{\min}^{\mathrm{U}}$. From [27] and [8],

$$\mu_x^\varepsilon = \frac{p_x^\varepsilon}{\sum_{z \in X} p_z^\varepsilon}, \quad \text{where} \quad p_x^\varepsilon = \sum_{T \in \mathbf{T}_x} \prod_{(y,z) \in T} P_{yz}^\varepsilon. \quad (20)$$

Then,

$$\mu_x^\varepsilon = \frac{\varepsilon^{\gamma^{\mathrm{L}}(x) - \gamma_{\min}^{\mathrm{U}}} \varepsilon^{-\gamma^{\mathrm{L}}(x)}}{\varepsilon^{-\gamma_{\min}^{\mathrm{U}}}} \cdot \frac{p_x^\varepsilon}{\sum_{z \in X} p_z^\varepsilon} \quad (21)$$

Since $\gamma^{\mathrm{L}}(x) > \gamma_{\min}^{\mathrm{U}}$, we know that $\lim_{\varepsilon \to 0^+} \varepsilon^{\gamma^{\mathrm{L}}(x) - \gamma_{\min}^{\mathrm{U}}} = 0$. Then it remains to show that $\varepsilon^{-\gamma^{\mathrm{L}}(x)} p_x^\varepsilon$ is bounded above, and $\varepsilon^{-\gamma_{\min}^{\mathrm{U}}} \sum_{z \in X} p_z^\varepsilon$ is bounded below. These bounding arguments follow similar techniques as the proof of Lemma 1 in [27] and are omitted for brevity. ■

Corollary 2: Define $X^\star := \{x \in X \,|\, \gamma^{\mathrm{L}}(x) \leq \gamma_{\min}^{\mathrm{U}}\}$. Then

$$\lim_{\varepsilon \to 0^+} \sum_{x \in X^\star} \mu_x^\varepsilon = 1 \quad (22)$$

V. Continuous state spaces

We now establish similar necessary conditions for stochastic stability of a partition in a compact set. Let $\Phi = \{\phi_n\}$ be a Markov chain on compact set X and let $P(x, A)$ be its transition kernel, for $A \in \mathcal{B}(X)$. The reader is referred to [20] for the necessary preliminaries.

Lemma 2: A ψ-irreducible Markov chain Φ on a compact set X admits a unique stationary measure ν.

Proof: Let Φ be a ψ-irreducible Markov chain on compact set X. Applying Theorem 8.2.5 from [20], Φ is either recurrent or transient. Seeking a contradiction, suppose Φ is transient. Then X is transient, so may be covered by a countable number of uniformly transient sets. Because X is compact, every cover has a finite subcover; thus, X may be covered by a finite number of uniformly transient sets, which is impossible. Therefore, Φ cannot be transient and must be recurrent, by Theorem 8.2.5 in [20]. Applying Theorem, 10.0.1 in [20], Φ must have a unique stationary measure. ■

A. Stochastic stability in continuous state spaces

As in Section III, we partition the continuous state space X into disjoint sets $\{X_i\}_{i\in L}$ and bound the probabilities of moving between the partitions. We begin with a simple generalization of Lemma 3.2 of Chapter 6 of [8].

Lemma 3: Let $P(x, X_j)$ be the transition kernel for a ψ-irreducible Markov chain Φ over a compact set X. Suppose X is partitioned into disjoint sets $\{X_i\}_{i\in L}$, with L finite and $\psi(X_i) > 0,\ \forall i \in L$. Suppose also that for all $i \neq j$ there exist $p_{ij}^{\mathrm{L}},\ p_{ij}^{\mathrm{U}} \geq 0$ such that the transition probabilities satisfy

$$p_{ij}^{\mathrm{L}} \leq P(x, X_j) \leq p_{ij}^{\mathrm{U}},\ \forall x \in X_i. \tag{23}$$

Define $\mathcal{G}^{\mathrm{L}} = (L, \mathcal{E}^{\mathrm{L}})$ such that $(i,j) \in \mathcal{E}^{\mathrm{L}}$ if and only if $p_{ij}^{\mathrm{L}} > 0$, and assign weight p_{ij}^{L} to edge (i,j). Let $\mathbf{T}_i^{\mathrm{L}}$ denote the set of all spanning trees of $\mathcal{G}^{\mathrm{L}}$ rooted at i,[1] and define

$$Q_i^{\mathrm{L}} := \sum_{T\in\mathbf{T}_i^{\mathrm{L}}} \prod_{(j,k)\in E_T} p_{jk}^{\mathrm{L}}.$$

Define $\mathcal{G}^{\mathrm{U}}$, $\mathbf{T}_i^{\mathrm{U}}$ and Q_i^{U} similarly with weights p_{ij}^{U}. Then, if ν is the stationary distribution of the Markov chain Φ,

$$\frac{Q_i^{\mathrm{L}}}{\sum_{j\in L} Q_j^{\mathrm{U}}} \leq \nu(X_i) \leq \frac{Q_i^{\mathrm{U}}}{\sum_{j\in L} Q_j^{\mathrm{L}}} \tag{24}$$

Proof: Φ is ψ-irreducible, so there exists $T \in \mathbf{T}_i^{\mathrm{L}}$ with $\prod_{(j,k)\in E_T} p_{jk}^{\mathrm{L}} > 0$. Then, $\nu(X_i) > 0\,\forall i$. This guarantees there exists $T \in \mathbf{T}_i^{\mathrm{U}}$ with $\prod_{(j,k)\in E_T} p_{jk}^{\mathrm{U}} > 0$ since $p_{ij}^{\mathrm{L}} \leq p_{ij}^{\mathrm{U}},\ \forall i,j$. Consider a Markov chain over L with

$$p_{ij} = \frac{1}{\nu(X_i)} \int_{X_i} \nu(dx) P(x, X_j). \tag{25}$$

The stationary distribution, $\{\mu_i\}_{i\in L}$, of this Markov chain is $\{\nu(X_i)\}_{i\in L}$. Using weights p_{ij}, define $\mathcal{G}$, $\mathbf{T}$ and Q_i in the same fashion as $\mathcal{G}^{\mathrm{L}}$, $\mathbf{T}^{\mathrm{L}}$ and Q_i^{L}. By condition (23),

$$p_{ij}^{\mathrm{L}} \leq p_{ij} \leq p_{ij}^{\mathrm{U}},\ \forall i,j$$

so it follows that

$$Q_i^{\mathrm{L}} \leq Q_i \leq Q_i^{\mathrm{U}}$$

for all $i \in L$ and hence

$$\frac{Q_i^{\mathrm{L}}}{\sum_{j\in L} Q_j^{\mathrm{U}}} \leq \frac{Q_i}{\sum_{j\in L} Q_j} \leq \frac{Q_i^{\mathrm{U}}}{\sum_{j\in L} Q_j^{\mathrm{L}}} \tag{26}$$

The result follows since, by Lemma 3.1 in Chapter 6 of [8],

$$\nu(X_i) = \frac{\sum_{T\in\mathbf{T}_i} \prod_{(j,k)\in E_T} p_{jk}}{\sum_{i'\in L}\sum_{T\in\mathbf{T}_{i'}} \prod_{(j,k)\in E_T} p_{jk}} = \frac{Q_i}{\sum_{j\in L} Q_j}.$$

■

We apply arguments of Section IV-B and Lemma 3 to consider stochastic stability of elements of partitions of X.

Theorem 4: Let $\{P^\varepsilon\}$ be the transition kernels for a family of ψ-irreducible Markov chains Φ^ε over a compact set X parameterized by $\varepsilon \in (0, a]$, $a > 0$. Suppose X is partitioned into disjoint sets $\{X_i\}_{i\in L}$, with L finite and $\psi(X_i) > 0,\ \forall i \in L$. Suppose also that for all $i \neq j$ there exist $C^{\mathrm{L}}(i,j)$, $r^{\mathrm{L}}(i,j)$, $C^{\mathrm{U}}(i,j)$, $r^{\mathrm{U}}(i,j) > 0$ such that the transition kernels P^ε satisfy

$$C^{\mathrm{U}}(i,j)\varepsilon^{r^{\mathrm{U}}(i,j)} \leq P^\varepsilon(x, X_j) \leq C^{\mathrm{L}}(i,j)\varepsilon^{r^{\mathrm{L}}(i,j)} \tag{27}$$

$\forall x \in X_i, \varepsilon \in (0, a)$. Define the lower and upper potentials, $\gamma^{\mathrm{L}}(X_i)$ and $\gamma^{\mathrm{U}}(X_i)$ as in Section IV, and define $\gamma_{\min}^{\mathrm{U}} := \min_{i\in\{1,\ldots,l\}} \gamma^{\mathrm{U}}(X_i)$. Then, for any $j \in \{1,\ldots,l\}$, if $\gamma^{\mathrm{L}}(X_j) > \gamma_{\min}^{\mathrm{U}}$,

$$\lim_{\varepsilon\to 0^+} \nu^\varepsilon(X_j) = 0, \tag{28}$$

where ν^ε is the stationary distribution of Φ^ε.

Proof: The proof is directly analogous to that of Theorem 3: for each ε use $p_{ij}^{\mathrm{L}} = C^{\mathrm{U}}(i,j)\varepsilon^{r^{\mathrm{U}}(i,j)}$, $p_{ij}^{\mathrm{U}} = C^{\mathrm{L}}(i,j)\varepsilon^{r^{\mathrm{L}}(i,j)}$, in Lemma 3; show that $\varepsilon^{-\gamma_j^{\mathrm{L}}} Q_j^{\mathrm{U}}$ is bounded above and $\varepsilon^{-\gamma_{\min}^{\mathrm{U}}} \sum_{i\in L} Q_i^{\mathrm{L}}$ below, so that $\nu(X_j) \to 0$. ■

VI. Illustrative example

The following example illustrates use of the coarse state space analysis of Section III. Suppose two collaborators, Bob and Alice, each have the opportunity to purchase a new computer on any given day, and may choose either a PC or an Apple computer. The following factors influence their decision: (i) collaboration is easier for both Bob and Alice when they use the same type of computer, and (ii) PC and Apple advertisements influence the value Bob and Alice places on each type of computer.

This scenario can be represented by a game G, played repeatedly over days $t = \{0, 1, 2, \ldots\}$ and consisting of:
(i) Players $N = \{\text{Bob}, \text{Alice}\}$
(ii) Action sets $\mathcal{A}_i = \{\text{Mac}, \text{PC}\},\ i \in N$, with $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$
(iii) State space which reflects the ads seen by Bob and Alice:

$$X = \begin{cases} x_1 & \text{Bob and Alice both see Mac ads} \\ x_2 & \text{Bob sees a Mac ad, Alice sees a PC add} \\ x_3 & \text{Bob sees a PC ad, Alice sees a Mac add} \\ x_4 & \text{Bob and Alice both see PC adds} \end{cases}$$

(iv) Payoff functions, $U_i : \mathcal{A} \times X \to \mathbb{R},\ i \in N$, shown in Table I, which depend on the ad seen and the other player's computer choice.
(v) Action invariant, aperiodic, and irreducible state transition function, $P_{x\to x'} = 1/4,\ \forall x \in X$. Independent of their current choices, Bob and Alice each see either a Mac or PC ad with probability 1/2 each day. Seeing an ad increases the payoff Bob or Alice associates with that type of computer.

Consider the mistakes model of [18], [23] in which players make "mistakes," i.e., suboptimal decisions, with a small probability that exponentially decreases with the potential payoff loss. We use this to model Bob and Alice's decision making processes as an RPP. Each day, one of the two is chosen at random to purchase a new computer if desired. With high probability, he or she matches the other's current choice, maximizing personal utility. With low probability, the suboptimal choice is made. When player $i \in \{\text{Bob}, \text{Alice}\}$ purchases a new computer for day $t+1$, the decision $a_i(t+1) \in \{\text{Mac}, \text{PC}\}$ is made probabilistically:

$$\Pr[a_i(t+1) = a_i] = \begin{cases} 1 - \varepsilon^{\Delta U_i} & \text{if } a_i = a_i^\star \\ \varepsilon^{\Delta U_i} & \text{if } a_i = a_i^0 \end{cases}$$

where

$$a_i^\star := \operatorname*{arg\,max}_{a_i\in\{\text{Mac},\text{PC}\}} U_i(a_i, a_{-i}(t), x(t))$$

$$a_i^0 := \operatorname*{arg\,min}_{a_i\in\{\text{Mac},\text{PC}\}} U_i(a_i, a_{-i}(t), x(t))$$

$$\Delta U_i := U_i(a_i^\star, a_{-i}(t), x(t)) - U_i(a_i^0, a_{-i}(t), x(t))$$

[1] The chain is ψ-irreducible and $\psi(X_i) > 0,\ \forall i \in L$, so lower bounds p_{ij}^{L} can be chosen so that $\forall i \in L$, $\mathcal{G}^{\mathrm{L}}$ contains a spanning tree with root i.

		Alice	
		Mac	PC
Bob	Mac	5,5	1,0
	PC	0,1	2,2

Payoffs in state x_1

		Alice	
		Mac	PC
Bob	Mac	5,3	1,1
	PC	0,0	2,4

Payoffs in state x_2

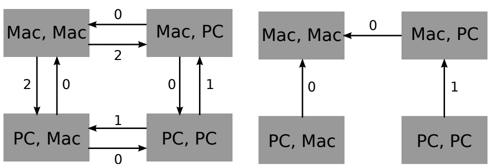
		Alice	
		Mac	PC
Bob	Mac	3,5	0,0
	PC	1,1	4,2

Payoffs in state x_3

		Alice	
		Mac	PC
Bob	Mac	3,3	1,0
	PC	0,1	4,4

Payoffs in state x_4

TABLE I: Numbers on the left and right represent row and column players payoffs respectively.



(a) Partition graph, $\mathcal{G}$ (b) Partition tree, $\mathcal{T}$

Fig. 1: Example graphs

Here $x(t) \in X$ and $a_{-i}(t)$ are the advertising state and the other player's computer choice at time t, respectively. The ensuing joint action is $(a_i(t+1), a_{-i}(t))$, and the state transitions with probability 1/4 to $x(t+1) \in \{x_1, x_2, x_3, x_4\}$, i.e., Bob and Alice each see either a Mac or Windows ad the next day with probability 1/2.

Resistances between states are given by

$$r([a,x],[(a_i', a_{-i}), x']) = \max_{a_i^\star \in \{a_i, a_i'\}} U_i(a_i^\star, a_{-i}, x) - U_i(a_i^0, a_{-i}, x), \quad (29)$$

$$r([a,x],[a,x']) = 0 \quad (30)$$

for $a \in \mathcal{A},\ x, x' \in X$. The equivalence relation

$$[a,x] \sim [a',x'] \iff a = a'.$$

defines a zero resistance partition. We refer to a partition by its action profile, $a \in \mathcal{A}$. The graph $\mathcal{G}$, defined by following the procedure of Section III-A and the minimum resistance tree rooted at $(\mathrm{Mac}, \mathrm{Mac})$ are shown in Figure VI.

Here, $\gamma(\mathrm{Mac}, \mathrm{Mac}) = 2$. No other partition is the root of an equal or lower resistance spanning tree of $\mathcal{G}$. Thus, $(\mathrm{Mac}, \mathrm{Mac})$ is the only stochastically stable partition. Corollary 1 implies that the only stochastically stable states are $[(\mathrm{Mac}, \mathrm{Mac}), x]$, $x \in X$. Using Corollary 1 allowed us to determine the stochastically stable states by computing trees over four states instead of sixteen.[2] Also note that if players only viewed PC adds, the unique stochastically stable state would be (PC,PC).

VII. Conclusion

We developed tools for analyzing stochastic stability in perturbed Markov chains. First, we showed that partitioning the state space in a certain way can lead to sufficient conditions for stochastic stability while alleviating the computational burden of computing solving for minimal weight spanning trees over the entire state space. Then, we gave necessary conditions for stochastic stability of states in an irregularly perturbed process. Finally, we established necessary conditions for stochastic stability of sets in a Markov chain over a continuous state space.

[2] Similar analysis can be used to reduce the necessary computation to determine stochastic stability in state based games such as those in [15].

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